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Discrete-time Replicator Equations and Evolutionary Dynamics of Learning in Optimal Transport Networks

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Abstract

The evolutionary dynamics of games, as systematically presented by Hofbauer and Sigmund [1], have broad applicability across numerous fields, including biology, ecology, economics, social systems, multi-agent frameworks, and learning theory. This interdisciplinary approach is rooted in applying dynamical systems theory to game theory, where various classes of dynamical systems – such as ordinary differential equations (e.g., the replicator equation), differential inclusions (e.g., best response dynamics), and reaction-diffusion models – serve as the mathematical backbone. A principal theme within this framework is the suboptimality of Nash equilibria in non-cooperative games concerning global system performance, quantitatively captured by the notion of the price of anarchy. This measure has received considerable interest in recent research.

In this talk, we first describe a novel application of replicator dynamics by formulating a discrete-time variant of continuous evolutionary dynamics. To this end, we introduce discrete-time replicator equations as a tool for analyzing optimal transport networks subject to congestion. Our presentation begins with the introduction of a Wardrop optimal network [2], which supports Wardrop equilibrium flows – flows that simultaneously satisfy Nash equilibrium conditions and achieve system-wide optimality; these networks are characterized by the price of anarchy equal to its least value, that is 1.

Building on this, we propose a dynamical model for optimal flow distribution in Wardrop optimal networks [2], leveraging the tools of evolutionary game theory, which places emphasis on the temporal evolution of strategies rather than static equilibria. Central to the analysis of evolutionary dynamics is the replicator equation, which quantifies the growth rate of agents adopting a specific strategy based on the deviation of the strategy's payoff from the population's average payoff. Strategies with above-average fitness increase in frequency, while those with below-average fitness decrease.

Our discrete-time dynamical model is based on mean-field replicator equations, defined on probability simplices, and generated by nonlinear, order-preserving mappings [3, 4]. In this talk, we focus on discrete-time replicator dynamical systems driven by Schur-convex potential functions [5], which will be demonstrated by using complete symmetric functions, gamma functions, and symmetric gauge functions as generators of the replicator dynamical systems. We study the system behavior, convergence, and stability properties, providing key insights into their temporal evolution. In particular, we examine the characteristics of Nash equilibria, the conditions for convergence to fixed points, and asymptotic stability within the replicator dynamics framework, utilizing tools from dynamical systems theory, such as Lyapunov functions. In the replicator systems under study, the Nash equilibrium, Wardrop equilibrium, and the system optimum converge to the same point in the state space, representing the flow distribution in the network.

We discuss the price of anarchy under different scenarios of transport network capacity and congestion parameters, along with the total costs at Nash equilibrium and system optimum. This analysis helps in understanding the dynamics of transportation networks and optimizing the learning rate for effective convergence to the Nash equilibrium. Our results demonstrate the significant impact of learning rates on the convergence behavior of replicator dynamics in transportation networks.

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To analyze the dynamics of learning using Poincare sections, we consider a multi-agent transportation system, where agents adapt their strategies over time based on the discrete-time replicator equations, both deterministic and stochastic. We present the bifurcation diagrams for evolutionary dynamics of learning in case of stochastic replicator equations. These diagrams provide valuable insights into the dynamic behavior of the stochastic replicator equations model. By analyzing these diagrams, one can understand how the system transitions between different regimes of stability and chaos as the learning parameter is varied, which is crucial for designing optimal transport networks, where learning dynamics play a significant role. We also present the simulation results that validate the theoretical analysis, including the examination of convergence rates for the orbits of the replicator system to fixed points under various function types generating the replicator dynamics.

Finally, we emphasize that potential applications of the proposed replicator dynamical model include applications to neural network models and the analysis of learning in neural network dynamics [6].

***K*eywords** Replicator equation · Evolutionary dynamics · Wardrop optimal network · Replicator dynamical system · Learning dynamics · Stability · Equilibrium · Optimum

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