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CONTINUITY OF SOLUTIONS OF PARAMETRIC OPTIMIZATION PROBLEMS

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ABSTRACT

Parametric optimization problems are tasks in which the objective function and/or the domain over which optimization occurs change depending on a parameter. The main subjects of study in these problems are the optimal value function and the multivalued mapping of the solutions to the optimization problem, which depend on the parameter.

Examining such problems is significant in the general theory of optimization and operations research due to their direct connection with the sensitivity and stability of optimization tasks under various disturbances. In practical problems, inaccuracies inevitably arise in observing a given process, as well as errors in the numerical processing of the obtained data. Therefore, it is necessary to know whether the obtained numerical solution to the disturbed optimization problem is close (in some sense) to the solution of the original problem.

Such tasks naturally arise in game theory and mathematical economics. In a standard setup, the optimal consumption of a given agent is sought depending on their income and market prices. Another typical situation is when an optimal investment strategy is considered as a function of market indices.

A key tool in parametric optimization problems is Berge's maximum theorem. This theorem provides a sufficient condition for the continuity of the optimal value function and the upper semicontinuity of the solutions to the optimization problem. One important application of this theorem is in obtaining equilibrium existence results in game theory.

A substantial part of the study of such problems involves working with multivalued mappings. The sets over which we optimize are given as values of a multivalued mapping with the parameter as its argument. The set of solutions is also a multivalued mapping that has the parameter as its argument. We are mainly interested in the continuity properties of these multivalued mappings, specifically in what assumptions to impose on the mapping and what conclusions we would like to obtain for the set of solutions.

The continuity of multivalued mappings is a significantly more complex and ambiguous concept than the continuity of functions. The definition and study of the notion of continuity initially belonged to theoretical mathematics. Continuously developing fields of applied mathematics, such as optimization, economics, and others, further stimulate the expansion of these constructions, addressing the need for such a mathematical apparatus through which broader concepts can be established and more global results can be derived. Analogous to the relationship between limits of sequences and limits of functions, theories of convergence of sequences of sets and continuity of multivalued mappings have developed in parallel. The concepts of internal and external limits of a sequence of sets were first introduced by the French mathematician Paul Painlevé (1863-1933) in 1902, with the convergence of a sequence of sets characterized by the coincidence of its internal and external limits. Later, after the first quarter of the 20th century, scholars like Felix Hausdorff (1868-1942)

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and Kazimierz Kuratowski (1896-1980) developed and popularized Painlevé's concept, basing their work on it. This is why today this convergence is known as Painlevé-Kuratowski convergence.

With sets as values, a multivalued mapping allows for a similar construction to be applied to it. Thus, the concept of Painlevé-Kuratowski continuity of a multivalued mapping is reached. Another construction, which combines well with the metric in a metric space, was introduced and studied by mathematicians Pompeiu (1873-1954) and Hausdorff.

Another type of continuity we will consider in our work aligns well with the topological structure of the space and can be defined for mappings between arbitrary topological spaces. In our work, we will call this type of continuity topological.

It is established that in a metric space with compact images, the three types of continuity of a multivalued mapping coincide. It is precisely compact images that are required by Berge's theorem. The relaxation of the compactness condition while preserving the conclusion in Berge's theorem is a question explored by various mathematicians. In a recent article by Feinberg et al., this problem is considered in general topological spaces.

In our study, we deal with this issue when the space over which we optimize is metric. Our main focus is proving the upper semicontinuity of the multivalued mapping of the solutions to an optimization problem. We also examine the interrelationships between different conditions imposed to relax compactness. We obtain several results that we have not encountered in the literature, summarizing and placing in a single framework known theorems in this field. In the beginning we optimize the objective function over a finite-dimensional space, with a weak type of continuity (Painlevé-Kuratowski) of the multivalued mapping additionally imposing a condition to restrict the behavior of the function at infinity. In a case where the objective function does not depend on the parameter, we introduce a new concept of well-posedness. We use it to obtain a general result for which we additionally require only the continuity of the multivalued mapping in the sense of Painlevé-Kuratowski and the continuity of the objective function. After this we provide sufficient conditions for the well-posedness of a pair. We show how known results are obtained as special cases. During our considerations, we introduce a new type of lower semicontinuity, stronger than all known to us so far.

When the objective function depends on the parameter, we obtain a theorem, which provides both new (to our knowledge) results and generalizes Berge's Theorem in its part concerning the semicontinuity of the solutions, as well as other previous results.

After this we obtain so-called inverse results. They address the question: given the upper semicontinuity of the solutions for all objective functions of a certain type, what can we say about the other objects in the problem, for example, the multivalued mapping. Most authors focus on proving the upper semicontinuity of the solutions. This is because, when we have upper semicontinuity of the mapping at some point, we know that for other points that are in some sense close to all solutions to the optimization problem at these points are close to solutions at but not to all solutions at. When we have lower semicontinuity of the mapping we know that at points close to the solutions to the optimization problem approximate well all solutions obtained at but there may also be those that are very far from the solutions at. Still, the preference is to have solutions close to those of the original problem though possibly not encompassing all its solutions, rather than having a good approximation of all solutions to the original problem but with the potential problem of some of the obtained solutions being very far from the solutions to the original problem.

***K*eywords** Optimization problems · Parametric problems · Objective function · Berge's theorem

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