FIBERED CATEGORIES: 2-GENERALIZED CROSSED MODULES

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ABSTRACT

Algebraic models of homotopy types are essential tools in homotopical algebra and higher category theory. In this context, 2-crossed modules were introduced by Conduché [3] as a categorical and algebraic model for connected homotopy 3-types, extending the classical notion of crossed modules that model homotopy 2-types.

The structure of a 2-crossed module consists of group homomorphisms equipped with compatible actions and Peiffer liftings, where the actions are typically given by conjugation. To extend the reach of this framework, the concept of a 2-generalized crossed module was defined in [7], allowing arbitrary group actions rather than restricting to conjugation. This generalization provides a more flexible algebraic setting for modeling higher-dimensional homotopical structures.

Let $\mathcal{F}$ and $\mathcal{Z}$ be two categories, and let $\Omega\colon \mathcal{F} \to \mathcal{Z}$ be a functor. We say that Ω is a category fibred over $\mathcal{Z}$ via Ω if and only if every morphism $\alpha\colon Z' \to Z$ in $\mathcal{Z}$ and every object $F \in \mathcal{F}$ such that $\Omega(F) = Z$, there exists a cartesian morphism $\beta\colon F' \to F$ in $\mathcal{F}$ such that $\Omega(\beta) = \alpha$. This condition allows us to "pullback" objects along morphisms in the base category, preserving structure.

The notion of fibered category was introduced by Grothendieck in [5] to formalize geometric ideas in descent theory by organizing objects and morphisms over a base category. This framework was later developed in detail by Bénabou [1], explaining its logical foundations and emphasized its role in category theory over a base with pullbacks. Fibered categories also offer a natural setting to study algebraic models of homotopical structures such as crossed modules, which play a key role in modeling homotopy 2-types [2]. Together, these concepts form essential tools in higher category theory and modern algebraic topology [1, 8].

The fibration of the category of 2-crossed modules over groups was investigated in [4], while fibrations and cofibrations of generalized crossed modules were introduced in [6].

In this work, it is shown that the category of 2-generalized crossed modules is a fibered category.

***K*eywords** Fibered category · Generalized crossed module · Pullback

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