AN EXTENSION OF MARKOV-TYPE OPERATORS. APPLICATIONS TO THE FRACTAL THEORY.

Ion Mierlus Mazilu[1,*], Lucian Niţă[2]

[1]*Technical University of Civil Engineering Bucharest, Romania*
[2]*Technical University of Civil Engineering Bucharest, Romania*

ABSTRACT

In this paper, we provide an extension of the concept of Markov-type operator. Classical Markov-type operators associated with iterated function systems have traditionally been studied either for scalar-valued measures or for vector measures taking values in a Hilbert space, a restriction that leaves out a substantial class of infinite-dimensional settings in which such operators arise naturally. Motivated by this gap, we consider the case when the Markov-type operator, associated to a family of contractions together with a compatible family of bounded linear operators, acts on Borel measures of bounded variation which take values in the conjugate of an arbitrary Banach space. This generalization is carried out uniformly for finite, countable, and uncountable families of contractions, thereby unifying constructions that are usually treated separately in the existing literature. We give the main properties of this operator, including its boundedness on the space of Banach dual-valued measures, explicit operator norm estimates in terms of the norms of the associated linear operators and the contraction ratios, and change-of-variable formulas relating the operator to vector-valued integration. These results are obtained with respect to a Monge-Kantorovich type norm defined on the space of vector measures, which provides the appropriate functional framework for studying continuity and convergence properties of the operator. Then, using the Markov-type operator, we construct a non-linear operator, which has, in some cases, a unique fixed point, named the fractal measure. Sufficient contractivity conditions, expressed through the operator norms and the contraction ratios of the underlying maps, are established under which the Banach Fixed Point Theorem guarantees the existence and uniqueness of this measure. One can find a connection between the fractal measure and the attractor, that is, the fractal set, associated to the family of contractions, showing that the support of the fractal measure coincides with this attractor in appropriate settings and thereby preserving the classical geometric interpretation inherited from the Hutchinson construction. We also give some examples, in both finite-dimensional and infinite-dimensional Banach spaces, to illustrate the theoretical facts and to demonstrate that the derived contractivity conditions can be verified explicitly in concrete situations.

Keywords Markov operator · attractor · fractal measure · iterated function system · Monge-Kantorovich norm

References

[1] Altomare, F., Montano, M. C., Leonessa, V., Raşa, I., On differential operators associated with Markov operators, J. Funct. Anal., vol. 266, no. 6 (2014), 3612-3631.

[2] Barnsley, M. F., Fractals everywhere, Academic Press, 2014.

[3] Barnsley, M. F., Demko, S., Iterated function systems and the global construction of fractals, Proc. R. Soc. Lond. A. Math. Phys. Sci., vol. 399, no. 1817 (1985), 243-275.

[4] Bartle, R.G., A general bilinear vector integral, Studia Math., vol. 15 (1956), 337-352.

[*]*Corresponding Author's E-mail: ion.mierlusmazilu@utcb.ro*

[5] Chiţescu, I., Spaţii de Funcţii, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1983.

[6] Chiţescu, I., Ioana, L., Miculescu, R., Niţă, L., Monge-Kantorovich norms on spaces of vector measures, Results in Mathematics 2016, 70, 349-371.

[7] Chiţescu, I., Ioana, L., Miculescu, R., Niţă, L., Sesquilinear uniform vector integral, Proc. Indian Acad. Sci. (Math Sci), vol. 125, no. 2 (2015), 187-198.

[8] Chiţescu, I., Niţă, L., Fractal vector measures, U.P.B. Sci. Bull. Series A, vol. 77, iss. 3 (2015), 219-228.

[9] D'Aniello, E., Steele, T.H., Attractors for classes of iterated function systems, Eur. J. Math. 2019, 5, 116-137.

[10] Diestel, J., Uhl, J.J., Vector Measures, Mathematical Surveys, no. 15, American Mathematical Society, Providence, RI, 1977.

[11] Dinculeanu, N., Vector Measures, VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.

[12] Dinculeanu, N., Vector Integration and Stochastic Integration in Banach Spaces, John Wiley & Sons, New York, Chichester, Weinheim, Brisbane, Singapore, Toronto, 2000.

[13] Elton, J. H., An ergodic theorem for iterated maps, Ergodic Theory and Dynamical Systems, vol. 7, no. 4 (1987), 481-488.

[14] Falconer, K.J., Fractal Geometry: Mathematical Foundations and Applications, John Wiley & Sons, Chichester, 1990.

[15] Feng, D.-J., Hu, H., Dimension theory of iterated function systems, Commun. Pure Appl. Math., vol. 62, no. 11 (2009), 1435-1500.

[16] Hutchinson, J. E., Fractals and self similarity, Indiana Univ. Math. J., vol. 30, no. 5 (1981), 713-747.

[17] Kigami, J., Analysis on Fractals, Cambridge University Press, no. 143, 2001.

[18] Kleinbock, D., Lindenstrauss, E., Weiss, B., On fractal measures and Diophantine approximation, Sel. Math., vol. 10, no. 4 (2005), 479.

[19] Klinga, P., Kwela, A., Staniszewski, M., Size of the set of attractors for iterated function systems, Chaos Solit. Fractals, vol. 128 (2019), 104-107.

[20] Léonard, C., From the Schrödinger problem to the Monge-Kantorovich problem, J. Funct. Anal., vol. 262, no. 4 (2012), 1879-1920.

[21] Leśniak, K., Snigireva, N., Strobin, F., Vince, A., Transition phenomena for the attractor of an iterated function system, Nonlinearity, vol. 35, no. 10 (2022), 5396.

[22] Mendivil, F., Vrscay, E.R., Self-affine vector measures and vector calculus on fractals, Fractals in Multimedia (2002), 137-155.

[23] Miclo, L., On hyperboundedness and spectrum of Markov operators, Invent. Math., vol. 200 (2015), 311-343.

[24] Mierluş-Mazilu, I., Niţă, L., About some Monge-Kantorovich type norms and their application to the theory of fractals, Mathematics 2022, 10, 1850076.

[25] Niţă, L., Fractal vector measures in the case of an uncountable iterated function system, RJMCS 2015, 5, 151-163.

[26] Secelean, N.A., Măsură şi fractali, Editura Universităţii Lucian Blaga, Sibiu, 2002.

[27] Sireţchi, G., Analiză Funcţională, Centrul de Multiplicare al Universităţii Bucureşti, 1982.

[28] Sireţchi, G., Spaţii Concrete în Analiza Funcţională, Centrul de Multiplicare al Universităţii Bucureşti, 1982.

[29] Strobin, F., On the existence of the Hutchinson measure for generalized iterated function systems, Qual. Theory Dyn. Syst., 2020, 19, 1-21.

[30] Villani, C., Optimal Transport: Old and New, Grundlehren der mathematischen Wissenschaften, vol. 338, Springer, Berlin, 2009.