NULL HYBRID (1,3)-BERTAND CURVES

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ABSTRACT

The concept of hybrid numbers, introduced by Özdemir [13], generalizes complex, hyperbolic, and dual numbers. A hybrid number is expressed as $H = h_0 + h_1\mathbf{i} + h_2\boldsymbol{\epsilon} + h_3\mathbf{h}$ where $h_0, h_1, h_2, h_3 \in \mathbb{R}$ and $\mathbf{i}\times\mathbf{i} = -1, \boldsymbol{\epsilon}\times\boldsymbol{\epsilon} = 0, \mathbf{h}\times\mathbf{h} = 1, \mathbf{i}\times\mathbf{h} = -\mathbf{h}\times\mathbf{i} = \boldsymbol{\epsilon}+\mathbf{i}$ are satisfied. The set of hybrid numbers is identified with the set of four-dimensional Minkowski space of index 2, $\mathbb{R}^4_2$, analogous to how the four-dimensional Euclidean space, $\mathbb{E}^4$, is identified with quaternions. Bharathi and Nagaraj [6] used quaternions to study the geometry of the curves in $\mathbb{E}^4$, and Akbıyık [1] extended this approach by using hybrid numbers to investigate non-null curves in $\mathbb{R}^4_2$.

Null curves are, on the other hand, of great importance in space-time geometry and physics. Consequently, extensive research has been done on null curves in Minkowski spaces of dimension n and index q. A comprehensive study on null curves in the (m+2)- dimensional manifolds of index q is provided by Duggal and Jin [8]. Alo [5] studied null hybrid curves and gave some properties of null hybrid Bertrand curves. They constructed a null Frenet frame for null hybrid curves using the null frame of the associated null spatial hybrid curve. They derived the null Frenet formulas and established relations between the curvatures of these two curves. Furthermore, they examined special null hybrid curves, null hybrid Bertrand curves, and showed that any such curve is a null spatial hybrid Bertrand curve- consistent with the fact that Bertrand curves in the space of dimension $n \geq 4$ are degenerate curves. This property of Bertrand curves in $\mathbb{E}^n$ $(n \geq 4)$ was initially proved by Matsuda and Yorozu [12], so they presented a new type of Bertrand curves in a space of dimension $n \geq 4$, called (1,3)-Bertrand curves. A curve $\hat{\Gamma}$ is said to be a (1,3)-Bertand mate of Γ if the normal plane spanned by the principal normal and the binormal of $\hat{\Gamma}$ coincides with the normal plan spanned by the principal normal and the binormal of Γ.

In this presentation, we investigate null hybrid (1,3)-Bertrand curves. We give conditions for a curve $\hat{\Gamma}$ to be a null hybrid (1,3)-Bertrand mate of the curve Γ. Furthermore, if γ is an associated null spatial hybrid curve of Γ, and $\hat{\gamma}$ a null Bertrand mate of γ, we examine the conditions under which the curve $\hat{\Gamma}$, whose associated curve is $\hat{\gamma}$, is a null hybrid (1,3)-Bertrand mate of Γ.

***K*eywords** Hybrid Numbers · Null Hybrid Curves · (1,3)-Bertrand Curves

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