

REVERSIBLE CANONICAL MEASURE FOR A ZERO-RANGE PROCESS WITH INDEPENDENT MARKOV NODE FAILURES

El'mira Yu. Kalimulina^{1,2,*}

¹Kharkevich Institute for Information Transmission Problems of the Russian Academy of Sciences (IITP RAS), Moscow, Russia

²Lomonosov Moscow State University (MSU), Moscow, Russia

ABSTRACT

We consider a conservative particle transport model on a network with independent Markov node failures: an edge is open only when both its endpoints are active.

Let $G_0 = (V, E)$ be a finite connected undirected graph, $|V| = N \geq 2$, with vertex degrees k_i , $i \in V$. The rates of the zero-range process (ZRP) are given by a function $u: \mathbb{Z}_+ \rightarrow [0, \infty)$ with $u(0) = 0$, $u(n) > 0$ for $n \geq 1$; set $p(0) := 1$, $p(m) := \prod_{n=1}^m u(n)^{-1}$ for $m \geq 1$. At each vertex an independent two-state Markov chain $I_v(t) \in \{0, 1\}$ is given with rates $\alpha > 0$ ($1 \rightarrow 0$) and $\beta > 0$ ($0 \rightarrow 1$); set $a := \beta/(\alpha + \beta)$, and let $\mu_a := \bigotimes_{v \in V} \text{Bernoulli}(a)$, which is reversible since $a\alpha = (1-a)\beta$. The random subgraph $G(t) := (V, \{\{i, j\} \in E : I_i(t)I_j(t) = 1\})$ is uniquely determined by $I(t)$. The joint Markov process $(\eta(t), I(t))$ on $\mathbb{Z}_+^V \times \{0, 1\}^V$ has generator

$$\mathcal{L}^{\text{tot}} f(\eta, I) = \sum_{i \in V} \sum_{j \sim i} I_i I_j \frac{u(\eta_i)}{k_i} [f(\eta^{i \rightarrow j}, I) - f(\eta, I)] + \mathcal{L}_I f,$$

where $\eta^{i \rightarrow j}$ denotes the configuration after a particle has moved from i to j (the term vanishes if $\eta_i = 0$ since $u(0) = 0$); \mathcal{L}_I is the generator of $|V|$ independent two-state chains.

Theorem 1. *Let $\{q_i\}_{i \in V}$ be positive weights; let $\{A_{ij}(I)\}_{i, j \in V}$ be symmetric conductances [3]: $A_{ij}(I) = A_{ji}(I) \geq 0$, $A_{ij} \equiv 0$ if $\{i, j\} \notin E$; let \mathcal{L}_I be the generator of an irreducible Markov chain on $\{0, 1\}^V$ with reversible measure μ , independent of η . Then for the generator*

$$\tilde{\mathcal{L}} f(\eta, I) = \sum_{i, j \in V} A_{ij}(I) \frac{u(\eta_i)}{q_i} [f(\eta^{i \rightarrow j}, I) - f(\eta, I)] + \mathcal{L}_I f$$

and any $M \in \mathbb{Z}_+$ the canonical measure

$$\pi_{N, M}(\eta, I) = \frac{1}{Z_{N, M}} \mu(I) \prod_{i \in V} p(\eta_i) q_i^{\eta_i} \cdot \mathbf{1} \left\{ \sum_{i \in V} \eta_i = M \right\}$$

is reversible with respect to $\tilde{\mathcal{L}}$ and unique on the sector $\{\sum_i \eta_i = M\}$ provided the joint chain is irreducible. In the principal example $q_i = k_i$ (the unnormalised invariant measure of the simple random walk on G_0), $A_{ij}(I) = I_i I_j \mathbf{1}_{\{i \sim j\}}$ and $\mu = \mu_a$; the joint chain is irreducible whenever G_0 is connected, $N \geq 2$ and $\alpha, \beta > 0$, and the reversible measure takes the explicit form

$$\pi_{N, M}(\eta, I) \propto \prod_{i \in V} p(\eta_i) k_i^{\eta_i} \cdot \prod_{i \in V} a^{I_i} (1-a)^{1-I_i} \cdot \mathbf{1} \left\{ \sum_i \eta_i = M \right\}.$$

*Corresponding Author's E-mail: eyk@iitp.ru

Consequences for the η -marginal. (i) In the principal example the η -marginal of $\pi_{N,M}$ coincides with the classical ZRP canonical measure [1], $\pi_{N,M}^\eta(\eta) \propto \prod_i p(\eta_i) k_i^{\eta_i} \mathbf{1}\{\sum \eta_i = M\}$, for all $\alpha, \beta > 0$ and without any limiting procedure in the activation rates; in particular, for any sequence of base graphs $\{G_N\}$ on which the failure-free ZRP admits a thermodynamic limit with critical density and condensation [2], symmetric independent node activity does not shift the static phase diagram nor relocate the condensate. (ii) In the regime $\alpha + \beta \rightarrow \infty$ with a fixed, standard averaging over the fast environment yields the generator $a^2 \mathcal{L}_\eta^{\text{ZRP}}$ for the η -projection: the dynamics is the failure-free ZRP slowed down by a factor a^{-2} . Within this reversible product-form class, a non-trivial shift of ρ_c is possible only through a modified single-site invariant measure, η -dependent failures, or a non-reversible edge dynamics with a different stationary structure.

Keywords zero-range process · reversible canonical measure · Markov node failures · condensation · random environment

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