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SOME RELATIONS INVOLVING ZEROS AND SPECIAL VALUES OF THE RIEMANN'S ZETA AND ALLIED FUNCTIONS

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ABSTRACT

Some recurrence relations for the Riemann zeta function $\zeta(s)$ at integer arguments, as well as relations involving nontrivial zeros of $\zeta(s)$ have been derived in [1]; for instance,

Proposition 1 ([1]) *For the Riemann zeta function $\zeta(s)$, it holds that*

$$\zeta(k) + \sum_{j=1}^{k-2} \lambda_j \zeta(k-j) + \gamma\lambda_{k-1} + k\lambda_k = 0, \quad (k \geq 2),$$

where λ_k are the coefficients in the Taylor series expansion of $\tilde{\Gamma}(z) = 1/\Gamma(1-z)$ around $z = 0$, and γ is the Euler-Mascheroni constant.

Proposition 2 ([1]) *For the nontrivial zeros ρ of the Riemann zeta function $\zeta(s)$,*

$$\sum_{j=0}^{k-2} \lambda_j \left(\sum_{\rho} \frac{1}{\rho^{k-j}} \right) + \left[\frac{1}{2}\gamma + 1 - \log(2\sqrt{\pi}) \right] \lambda_{k-1} + k\lambda_k = 0, \quad (k \geq 2),$$

where λ_k are the coefficients in the Taylor series expansion of the Riemann ξ-function (or completed zeta function) around $s = 0$, and γ is the Euler-Mascheroni constant.

The above propositions have been obtained using several formulas that were derived for the class of entire and meromorphic functions and that relate the sums of the nth powers of the reciprocals of zeros and poles of these functions with the coefficients of their Taylor series expansions [1], e.g.

Theorem 3 ([1]) *Let $f(z)$ be an entire function of finite order ρ, for which $p = \lfloor \rho \rfloor$, where p is the genus of f. Further, suppose that $\{a_n\}_{n\geq 1}$ is the sequence of zeros of $f(z)$. Then*

$$\sum_{j=0}^{k-p-1} \lambda_j \sigma_{k-j} + k\lambda_k = \sum_{j=k-p}^{k-1} \lambda_j (k-j) q_{k-j}, \quad k > p$$

where λ_j are the coefficients in the Taylor series expansion of $f(z)$ at $z = 0$, q_k are the coefficients of the polynomial in the Hadamard factorization of f, and σ_k are the sums of the form $\sigma_k = \sum_{n=1}^{\infty} 1/a_n^k$. Moreover, the above assertion holds true when ρ is not an integer.

The aim of this talk is to discuss other recurrence formulas involving Riemann zeta function and some other allied functions [2]. In particular, similar results are established for the digamma function, Barnes G-function (or double gamma function), and its logarithmic derivative [3], which will be based on the results of the paper [1] and the Weierstrass infinite product representations of entire functions [3, 4]. A recurrence formula for the values of the Riemann zeta function at odd positive integers is also derived. I will also discuss the relations for the multiple gamma functions [5] and

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Vignéras' multiple gamma functions [6]. The determinantal formulas for the sums of the nth powers of the reciprocals of zeros of entire or meromoprhic functions as well as for the coefficients λ_k will also be presented. I will conclude by discussing some further research in this direction and outlining possible applications of the same ideas and the results obtained.

***K*eywords** Zeta functions · Sums of powers of reciprocals of zero · Entire functions · Weierstrass infinite product representations · Recurrence formulas

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