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ISOMONODROMIC CONNECTIONS ON PRINCIPAL BUNDLES AND DYNAMICAL SYSTEMS

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ABSTRACT

A principal G-bundle P over a manifold M consists of a total space P, a base space M, a projection $\pi : P \to M$, and a free right action of a Lie group G on P that preserves the fibers of π. These geometric objects provide a natural framework for studying systems with internal symmetries [1]. In this context, connections on principal bundles represent geometric structures that enable horizontal lifting of curves from the base space to the total space, facilitating the study of parallel transport and holonomy. Logarithmic connections, a specific class of connections with simple poles at designated points, emerge in the study of differential equations with regular singular points and provide a rich geometric structure for analyzing control systems. The monodromy of a logarithmic connection, which measures the holonomy around loops encircling the singular points, serves as a topological invariant that characterizes the global behavior of the connection.

Isomonodromic deformations constitute particularly interesting families of connections, where the monodromy representation remains constant as the connection varies. These deformations are governed by the Schlesinger equations, a system of non-linear partial differential equations that describe how the residues of the connection must evolve to preserve the monodromy [5].

Control systems on manifolds represent a fundamental area of study in control theory, with applications ranging from robotics to aerospace engineering. The geometric approach to control theory, pioneered by [4], applies differential and algebraic geometry for understanding controllability, stability, and optimality of trajectories. When the configuration space possesses non-trivial topology, as is often the case for mechanical systems with constraints or symmetries [2], the geometric perspective becomes essential for a comprehensive analysis.

The main contribution of this work is the establishment of a precise correspondence between isomonodromic deformations of logarithmic connections on principal G-bundles and control systems on Riemann surfaces. This correspondence illuminates the geometric structures underlying controllability and optimality of trajectories, leading to several key insights. First, the monodromy-curvature correspondence serves to show how the curvature of a pulled-back connection relates to the residues of the original logarithmic connection. This result provides a bridge between the local differential structure of the connection and its global topological properties, allowing for a more comprehensive understanding of the system's behavior. Second, the structure of isomonodromic deformations is characterized through a non-linear partial differential equation satisfied by the map from the Riemann surface to the homogeneous space G/H, where H is a maximal compact subgroup of G. This equation encodes the evolution of the connection as the singular points move, providing a concrete description of how the geometry changes during the deformation.

The central theorem establishes a precise relationship between isomonodromic families of logarithmic connections and G-equivariant control systems on the total space of the principal bundle. This theorem demonstrates that the controllability of the system is equivalent to the condition that the

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residues of the connection generate the full Lie algebra $\mathfrak{g}$ under the Lie bracket operation. Furthermore, the trajectories of the control system project to geodesics on the Riemann surface with respect to a metric determined by the monodromy data, and the optimal control functions minimizing a quadratic cost functional correspond precisely to isomonodromic deformations of the connection. As explored by [3] in a work on nonholonomic mechanics, systems with constraints require geometric techniques for effective control. Thus, the approach developed here provides a systematic method for designing robust controllers that account for the geometric complexities of the task space, ensuring optimal performance even in challenging environments.

This geometric framework is extended to control systems with non-holonomic constraints, establishing a representation in terms of families of logarithmic connections where the singular points encode the geometric properties of the constraint distribution. This extension provides a tool for analyzing systems with velocity constraints, such as wheeled robots or underwater vehicles. This has implications beyond robotic control, extending to areas such as quantum control theory, where the geometric phases become crucial for understanding the evolution of quantum systems. The connection with integrable systems, emphasized by [6], suggests potential applications to machine learning algorithms for control on manifolds, where the preservation of geometric structures could lead to more efficient and robust learning processes.

Finally, a detailed computational example for $\mathrm{SU}(2)$ connections over a hyperbolic Riemann surface of genus 2 is provided, illustrating the applicability of the geometric framework. The example considers the specific case of controlling the orientation of a robot moving on a curved surface, demonstrating how the abstract theory translates into concrete control strategies. It begins with the choice of three singular points in the upper half-plane and the definition of residues that form a basis for the Lie algebra $\mathfrak{su}(2)$, ensuring controllability of the system. The computation of the map from the Riemann surface to the homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1)$ and the associated metric provides insight into the geometric structure induced by the connection. The geodesics with respect to this metric correspond to optimal trajectories for the control system, minimizing a quadratic cost functional related to the energy expenditure of the control inputs. The resulting control strategy guides the robot along a nearly vertical path in the upper half-plane while optimally adjusting its orientation, with robustness to perturbations ensured by the isomonodromic property of the connection. The robustness of this control strategy stems from the isomonodromic property: small perturbations of the path preserve the monodromy representation, ensuring that the system can recover from disturbances without significant deviations from the optimal trajectory. This property is particularly remarkable for robotic systems operating in uncertain environments or on surfaces with varying curvature, as discussed by Marsden and Ratiu [7].

***K*eywords** Isomonodromic connections · Principal bundles · Dynamical systems · Geometric control theory

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