Some Properties on Bidimensional Tribonacci numbers

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Abstract

The Tribonacci sequence, denoted by $\{t_n\}_{n\geq 0}$ and catalogued as A000073* in the OEIS [4], offers fertile ground for new and challenging investigations. Studied and popularised by Feinberg [1], this sequence is defined by the three-term linear recurrence

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \quad \text{for all integers } n \geq 3,$$

with initial terms $t_0 = 0$ and $t_1 = t_2 = 1$.

In this work, we explore a two-dimensional generalisation of the Tribonacci sequence, referred to as the bidimensional Tribonacci sequence and denoted by $\{T_{m,n}\}_{m,n\geq 0}$. Building upon the classical one-dimensional case, the bidimensional sequence was defined by Pethe [3] through a natural extension of the Tribonacci recurrence to two indices:

$$\begin{aligned} T_{(m,n)} &:= T_{(m-1,n)} + T_{(m-2,n)} + T_{(m-3,n)}, \quad && \text{for all } n \text{ and } m \geq 3 \\ T_{(m,n)} &:= T_{(m,n-1)} + T_{(m,n-2)} + T_{(m,n-3)}, \quad && \text{for all } m \text{ and } n \geq 3 \end{aligned}$$

with initial conditions involving complex values:

$$\begin{array}{lll} T_{(0,0)} = 0\,, & T_{(1,0)} = 1\,, & T_{(2,0)} = 1\,, \\ T_{(0,1)} = i\,, & T_{(1,1)} = 1 + i\,, & T_{(2,1)} = 1 + i2\,, \\ T_{(0,2)} = i\,, & T_{(1,2)} = 2 + i\,, & T_{(2,2)} = 2 + i2\,, \end{array}$$

where $i^2 = -1$ is the imaginary unit.

Our study extends some known results from the classical Tribonacci sequence (see for instance [2]) into the bidimensional context. In particular, we investigate algebraic properties, recurrence structures, and symmetries of this extension and derive new identities. Special attention is given to the role of complex values in the initial conditions and their impact on the emerging structure. This generalisation not only enriches the theoretical landscape surrounding recurrence relations but also provides a framework for further exploration of multidimensional recurrence relations.

***K*eywords** Tribonnaci numbers · Bidimensional Tribonnaci numbers · Bidimensional recurrence relations

References

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