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# GENERALIZED FOCAL CURVES OF HELICAL CURVES

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## ABSTRACT

In the work [11] published in 2005, Uribe-Vargas gave the notions of focal curve and focal curvatures of a Frenet curve in $m+1$-dimensional Euclidean space $\mathbb{E}^{m+1}$, $m \geqq 2$. For any spherical curve, the focal curve degenerates at the center of the sphere that contains the curve. That is why we study generalized focal curves of spherical curves. We investigate the relations between the Frenet frames and the differential-geometric invariants of a helical curve in three-dimensional Euclidean space $\mathbb{E}^3$ and its closely related curve in four-dimensional Euclidean space $\mathbb{E}^4$. To obtain the parametric representation of the generalized focal curve of a helical curve, the focal curve of the constructed four-dimensional curve is used. Only helical curves, also known as curves with constant slope, will be examined in this work. The case of non-helical curves is studied in [4]. We construct a new curve in $\mathbb{E}^4$ (4D-curve) that is associative to a given space curve in $\mathbb{E}^3$. Then we find the focal curve of the corresponding 4D-curve. The orthogonal projection of the obtained curve in $\mathbb{E}^3$ is called a generalized focal curve of the initial curve. We provide a few helical spherical curve examples to demonstrate the results that were obtained.

Let $I \subseteq \mathbb{R}$ be a zero-containing interval, and let $\boldsymbol{\alpha} : I \longrightarrow \mathbb{E}^3$ be a helical Frenet curve of class $C^4$ with an arc-length parametrisation and a parametrical equation

$$\boldsymbol{\alpha}(s) = \big(x(s),\, y(s),\, a.s\big)^T, \quad s \in I,\, a = const,\, a \in (-1,0) \cup (0,1).$$

The unit-speed curve $\boldsymbol{\gamma}$ that is closely related to $\boldsymbol{\alpha}$ were studied in [9] and has a parametric equation

$$\boldsymbol{\gamma}(s) = \big(x(s),\, y(s),\, \cos(a.s),\, \sin(a.s)\big)^T, \quad s \in I.$$

The curve $\gamma$ is a well-defined non-spherical Frenet curve with Frenet frames, a focal curve, and focal curvatures if $\alpha$ is a spherical space curve.

***K*eywords** Frenet curves · Helical curves · Focal curves · Spherical curves

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