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THE TRANSPOSE-INVERSE INVOLUTION ON $\mathrm{SL}(n, \mathbb{C})$-HIGGS BUNDLE MODULI SPACES AND ELECTROMAGNETIC DUALITY

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ABSTRACT

In this research, we investigate the geometric and physical implications of outer automorphisms acting on the moduli space of $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles over a compact Riemann surface of genus $g \geq 2$. The study of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories has been influenced by the Hitchin-Donagi-Witten correspondence [2, 5], which establishes a precise correspondence between the physics of these theories compactified on a Riemann surface and the geometry of Higgs bundle moduli spaces. In this framework, electromagnetic dualities—transformations that interchange electric and magnetic charges while preserving the physical content—manifest as automorphisms of the Hitchin integrable system. While the classical S-duality of Seiberg-Witten theory [7, 8] has been extensively studied, the role of outer automorphisms of the gauge group remains largely unexplored from a geometric perspective. For the group $\mathrm{SL}(n, \mathbb{C})$, the outer automorphism group is $\mathrm{Out}(\mathrm{SL}(n, \mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the transpose-inverse automorphism $g \mapsto (g^t)^{-1}$, which acts on the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ by $X \mapsto -X^t$. This automorphism induces a natural involution on the moduli space $\mathcal{M}_{\mathrm{SL}(n)}$ of semistable $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles, defined by $\tau(E, \varphi) = (E^*, -\varphi^t)$, where E^* denotes the dual bundle and φ^t is the transpose of the Higgs field with respect to the natural pairing.

The main result establishes that this involution τ preserves the hyperkähler structure on $\mathcal{M}_{\mathrm{SL}(n)}$ and acts in a highly structured way on the Hitchin fibration $\pi\colon \mathcal{M}_{\mathrm{SL}(n)} \to \mathcal{B}_{\mathrm{SL}(n)}$. The Hitchin fibration is a proper map whose base $\mathcal{B}_{\mathrm{SL}(n)}$ parametrizes the space of characteristic polynomials and whose generic fibers are abelian varieties encoding the electromagnetic charge lattice. We prove that τ acts on the base by reversing the signs of odd-degree invariant polynomials while preserving even-degree ones, and on the homology $H_1(A_u, \mathbb{Z})$ of a generic fiber A_u by multiplication by the sign $\varepsilon(n) = (-1)^{\lfloor (n-1)/2 \rfloor}$. This sign exhibits a striking periodicity modulo 4: it equals $+1$ when $n \equiv 0, 1 \pmod 4$ and -1 when $n \equiv 2, 3 \pmod 4$. The proof involves analyzing how the transpose operation interacts with the spectral curve construction developed by Beauville-Narasimhan-Ramanan [1] and understanding the induced map on Prym varieties following Mumford's classical theory [6].

Translating these results to physics via the Hitchin-Donagi-Witten correspondence yields a new electromagnetic duality. In the physical theory, BPS states are characterized by electromagnetic charges $\gamma \in H_1(A_u, \mathbb{Z})$, and their multiplicities $\Omega_u(\gamma)$ encode the spectrum of stable particles at a given vacuum labeled by $u \in \mathcal{B}_{\mathrm{SL}(n)}$. The geometric involution induces a physical duality relation of the form $\Omega_u(\gamma) = \Omega_{\sigma_n(u)}(\varepsilon(n) \cdot \gamma)$, where σ_n denotes the induced map on the Coulomb branch. The qualitative nature of this duality depends dramatically on the rank modulo 4. For $n = 2$, corresponding to the classic $\mathrm{SU}(2)$ Seiberg-Witten theory, the duality becomes trivial as both σ_2 and the charge transformation reduce to the identity, imposing no constraints beyond the standard S-duality. For $n = 3$ and $n = 4$, the duality manifests as a charge conjugation symmetry that relates BPS spectra at different points of the Coulomb branch by reversing electromagnetic charges,

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providing non-trivial constraints on the wall-crossing behavior studied by Gaiotto-Moore-Neitzke [3, 4]. For $n = 5, 6$ and higher ranks in the same congruence classes, the duality preserves charges while relating physically distinct vacua, imposing equality constraints on BPS multiplicities that must hold throughout wall-crossing transitions.

Thus, this work reveals that outer automorphisms provide a systematic source of new dualities whose properties depend on subtle number-theoretic patterns in the rank, and opens numerous directions for future investigation including connections to the geometric Langlands program, and applications to mirror symmetry and derived categories.

***K*eywords** Hitchin fibration · Higgs bundles · Outer automorphisms · BPS spectrum · Electromagnetic duality

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