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ANALYTICAL CHARACTERIZATION OF FIXED-POINT LOCI IN HIGGS BUNDLE MODULI SPACES VIA L^2-BOUNDS AND LAGRANGIAN GEOMETRY

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ABSTRACT

This work addresses the geometric study of involutions acting on moduli spaces of Higgs bundles over Riemann surfaces, with particular focus on the hyperelliptic involution in the genus-2 case. Higgs bundles, introduced by Hitchin in his seminal work [3], consist of pairs (V, Φ) formed by a holomorphic vector bundle V and a Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}(V) \otimes K_\Sigma)$ satisfying semistability conditions. The moduli space $\mathcal{M}(r, d)$ of semistable rank-r degree-d Higgs bundles carries a natural hyperkähler structure, and automorphisms of the underlying curve induce automorphisms of this moduli space by taking the pull-back [1, 2].

Every genus-2 Riemann surface Σ is hyperelliptic and admits a canonical involution $\iota\colon \Sigma \to \Sigma$ characterized by a degree-2 map $\pi\colon \Sigma \to \mathbb{P}^1$ to the Riemann sphere, branched at six Weierstrass points. This involution can be explicitly described in the affine model $\Sigma = \{(x, y) \in \mathbb{C}^2 : y^2 = f(x)\}$ where $f(x) = \prod_{i=1}^{6}(x - \lambda_i)$, as $\iota(x, y) = (x, -y)$. The hyperelliptic involution induces an automorphism $\iota^*\colon \mathcal{M}(r, d) \to \mathcal{M}(r, d)$ by pullback of both the vector bundle and the Higgs field. The general theory of real structures on Higgs bundle moduli spaces, developed by Schaposnik [5], establishes that the fixed-point locus $\mathcal{M}(r, d)^\iota$ of such involutions forms a Lagrangian submanifold with respect to the natural symplectic structure on the moduli space.

Building upon this result, our main contribution provides explicit quantitative bounds relating the hyperkähler distance between a Higgs bundle and its pullback to analytical invariants of the Higgs field. We focus on the moduli space $\mathcal{M}(2, 0)$ of rank-2 degree-0 semistable Higgs bundles over genus-2 curves, which forms a quasi-projective variety by the work of Nitsure [4]. Specifically, we establish two-sided inequalities of the form

$$\|\Delta(\Phi)\|_{L^2} \leq d_{\mathrm{hyp}}((V, \Phi), \iota^*(V, \Phi)) \leq C \cdot \|\Delta(\Phi)\|_{L^2},$$

where $\Delta(\Phi) = \Phi - \sigma^{-1} \circ \iota^* \Phi \circ \sigma$ represents the difference between the Higgs field and its transformed version under the involution, with $\sigma\colon V \to \iota^* V$ being an isomorphism identifying the bundle with its pullback. The universal constant $C = \sqrt{1 + K^2}$ is determined by the spectral properties of the linearized Hitchin equations. These bounds provide a concrete analytical criterion for measuring proximity to the fixed-point locus, establishing that a Higgs bundle lies in $\mathcal{M}(2, 0)^\iota$ if and only if $\|\Delta(\Phi)\|_{L^2} = 0$.

The techniques employed combine methods from differential geometry, including the analysis of hyperkähler metrics and associated Kähler forms $\omega_I, \omega_J, \omega_K$ satisfying the quaternionic relations, with moduli-theoretic tools and the analytical study of elliptic operators arising from the Hitchin equations. The proof relies on establishing comparison estimates between the hyperkähler distance and suitable Sobolev norms, then relating these to the L^2-norm through elliptic regularity theory.

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We illustrate our results with explicit computations on a model hyperelliptic curve with branch points at $\{0, 1, 2, 3, 4, 5, \infty\}$, computing distances and norms for various representative Higgs bundles. These examples include the zero Higgs field $(V, 0)$ which lies in the fixed-point locus with $d_{\text{hyp}} = 0$, off-diagonal configurations with

$$\Phi = \begin{pmatrix} 0 & \omega_1 \\ \omega_1 & 0 \end{pmatrix}$$

yielding $\|\Delta(\Phi)\|_{L^2} = 2\sqrt{2}$, and general diagonal fields demonstrating how the distance scales with the coefficients in the expansion along the basis of holomorphic differentials.

Our work opens several directions for future investigation, including extensions to higher genus and rank, applications to mirror symmetry where Lagrangian submanifolds play a fundamental role as branes in the A-model, and connections to mathematical physics where fixed-point loci appear naturally in gauge theories with orientifold planes and supersymmetric field theories on non-orientable surfaces.

Keywords Higgs bundles · Hyperelliptic involution · Lagrangian manifolds · Hyperkähler geometry · Moduli spaces

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