# On Generalised Leonardo Numbers

Carlos M. da Fonseca[1], Can Kızılateş[2], Paulo Saraiva[3,*], Anthony G. Shannon[4].

[1] *Kuwait College of Science and Technology, Kuwait*
[2] *Zonguldak Bulent Ecevıt Unıversıty, Department of Mathematics, Turkey*
[3] *Faculty of Economics, CMUC and CeBER, University of Coimbra, Portugal*
[3] *Warrane College, University of New South Wales, Australia*

## Abstract

The Leonardo sequence $(\mathrm{Le}_n)$ is defined by the inhomogeneous recurrence relation

$$\mathrm{Le}_n = \mathrm{Le}_{n-1} + \mathrm{Le}_{n-2} + 1\,, \quad \text{for } n \geqslant 2, \tag{1}$$

with initial conditions

$$\mathrm{Le}_0 = \mathrm{Le}_1 = 1\,. \tag{2}$$

Alternatively, it can be defined by the homogeneous recurrence relation

$$\mathrm{Le}_n = 2\,\mathrm{Le}_{n-1} - \mathrm{Le}_{n-3}\,, \quad \text{for } n \geqslant 3, \tag{3}$$

where in this case $\mathrm{Le}_0 = \mathrm{Le}_1 = 1$ and $\mathrm{Le}_2 = 3$ are the initial conditions. As with so many mathematical concepts, it is not always easy to establish when Leonardo numbers were first defined. This sequence has a *sui generis* history: first we found its extensions and more recently its interest has been reborn through very particular cases of them.

This talk covers some of the history of Leonardo numbers. We retrieve some of the most recent results on this sequence, as well as some relevant historical interconnections. Next, we present a matricial approach based on the determinant of certain Hessenberg matrices to finding the generating function of homogeneous recurrence relations, applying it in some examples after converting the inhomogeneous initial to the pertinent homogeneous form. In the end, we also provide some conjectures and open problems for some of its extensions involving the modular periodicity.

***K*eywords** Leonardo sequence · recurrence relations · Hessenberg matrices · determinant

## References

[1] P.R.J. Asveld, A family of Fibonacci-like sequences, Fibonacci Quart. 25 (1987), 81-83.

[2] E.W. Dijkstra, Smoothsort, an alternative for sorting in situ, Sci. Comput. Program. 1 (1982), 223-233.

[3] E.W. Dijkstra, Fibonacci numbers and Leonardo numbers, Archive EWD 797, www.cs.utexas.edu /users/EWD/ewd07xx/ EWD797.PDF

[4] C.M. da Fonseca, The generating function of a bi-periodic Leonardo sequence, Armen. J. Math. 16 (2024), 1-8.

[5] C.M. da Fonseca, P. Saraiva, A.G. Shannon, Revisiting some $r$-Fibonacci sequences and Hessenberg matrices, Notes Number Theory Discrete Math. 30 (2024), 704-715.

[6] C.M. da Fonseca, C. Kızılateş, P. Saraiva, A. G. Shannon, Generalised Leonardo numbers, Logic Journal of the IGPL (2025), jzaf005, https://doi.org/10.1093/jigpal/jzaf005.

[7] D. Singmaster, Some counterexamples and problems on linear recurrence relations, Fibonacci Quart. 8 (1970), 264-267, 279.

[*] *Corresponding Author's E-mail: psaraiva@fe.uc.pt*