IMPROVED COMPUTATIONAL TECHNIQUES FOR HEAT SOURCES LOCALIZATION

Simonetta BORIA[1], Nadaniela EGIDI[1], Lorella FATONE[1], Josephin GIACOMINI[1], Pierluigi MAPONI[1], Riccardo PIOMBIN[1,*],

[1] *School of Science and Technology, University of Camerino (UNICAM), Via Madonna delle Carceri 9, 62032 Camerino (MC), Italy*

ABSTRACT

We consider the inverse heat source problem, which consists in determining the position and intensity of unknown heat sources within a given domain based on observed temperature distributions. The inverse heat source problem has demonstrated significant potential for applications in fields such as thermal management [1], pollution detection [2], and in general in different engineering problems [3]. In particular, we are interested in the numerical solution of this problem for an application in the living sector [4], so the domain under consideration is a three-dimensional region. In this context, we propose a novel approach that leverages Green's function methods to represent the solution of the heat conduction problem under study. This, in turn, allows us to construct a Volterra equation of the first kind, to formulate the inverse problem.
The formulation of the representation formula is based on the solution of the following one-dimensional problem for the heat equation with appropriate boundary conditions and initial temperature distribution:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \alpha \frac{\partial u}{\partial x^2}(x,t) = f(x,t), & x \in \Omega,\ t \in [0,T], \\ \frac{\partial u}{\partial x}(0,t) = 0, & t \in [0,T], \\ \frac{\partial u}{\partial x}(L,t) = 0, & t \in [0,T], \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where we consider $\Omega = [0, L]$ as the spatial domain and $[0, T] \subset \mathbb{R}$ as the time interval; $\alpha \in \mathbb{R}$, $\alpha > 0$, is a constant term representing the thermal diffusivity, $u_0(x)$ is the function describing the initial temperature, i.e., the temperature at time $t = 0$, and $f(x,t)$ is the function representing the source term. The source term is modeled as a sum of Gaussian functions, which provided computational advantages in our case study, [5]. From standard arguments on heat equation theory, the solution of problem (1) can be written in integral form by involving the Green function associated with problem (1), see [6] for details.
From the solution of problem (1), we extend our methodology to the three-dimensional problem:

$$\begin{cases} \frac{\partial u}{\partial t}(\mathbf{x},t) - \alpha \Delta u(\mathbf{x},t) = \phi(\mathbf{x},t), & \mathbf{x} \in B,\ t \in [0,T], \\ \frac{\partial u}{\partial \hat{n}}(\mathbf{x},t) = 0, & \mathbf{x} \in \partial B,\ t \in [0,T], \\ u(\mathbf{x},0) = U_0(\mathbf{x}), & \mathbf{x} \in B; \end{cases} \tag{2}$$

where the spatial domain is $B = [0, L_x] \times [0, L_y] \times [0, L_z] \subset \mathbb{R}^3$, $L_x > 0$, $L_y > 0$, $L_z > 0$, and ∂B is the boundary of the domain B. Moreover, $\hat{n}(\mathbf{x})$ represents the unit outward normal at $\mathbf{x} \in \partial B$, Δ is the Laplacian operator, $\phi(\mathbf{x},t)$ is the three-dimensional function describing the source term and $U_0(\mathbf{x})$ is the function modeling the initial temperature. An integral formulation of the solution $u(\mathbf{x}, t; U_0, \phi)$ of problem (2) is obtained [4]. This involves the functions $U_0(\mathbf{x})$, $\phi(\mathbf{x},t)$ and the Green function associated to problem (2).
We study the following inverse problem: determine the source intensities that match some temperature measurements at specific points within the domain at precise different times. More specifically,

[*] *Corresponding Author's E-mail: riccardo.piombin@unicam.it*

given two positive integers I and J, we define $\mu_i(t_j)$,, $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J + 1$, as the temperature measurements at known points $\mathbf{x}_i \in B$, $i = 1, 2, \ldots, I$ at times t_j, $j = 1, 2, \ldots, J + 1$, we want to compute the source intensity $\phi(\mathbf{x}, t)$ such that $u(\mathbf{x}_i, t_j; U_0, \phi) = \mu_i(t_j)$, $i = 1, 2, \ldots, I$, $j = 1, 2, \ldots, J+1$. Supposing to know U_0, this is an integral equation for the source intensity function. We present some results obtained with an improved numerical solution of this integral equation. The main novelties are related to the use of the midpoint rule [7] for the numerical quadrature and a proper use of the measurements in different times to obtain a more efficient and stable discretization scheme. More precisely, the numerical quadrature leads to the definition of a linear system that, before being numerically solved, is modified by applying the Tikhonov regularization technique [8] to find a stable solution with non-negative components through a constrained minimization problem. Moreover, an iterative approach leads to solving the problem for subsequent subintervals $[0, t_j]$ of $[0, T]$ instead of attempting to solve it for the entire time interval at once. In particular, for a given $p < J + 1$, at the generic step k of the proposed iterative procedure we utilize the values of source intensities already computed at times t_j, $j = 1, \ldots, j_{k-p}$, U_0, ϕ, to compute the unknown values of the source intensities at times t_j, $j = j_{k-p}, \ldots, j_k$. So, we retain as variables only the source at the last p measurement times. Finally, some numerical results will be presented to illustrate the efficacy of our proposed method.

Keywords Inverse source problem · Volterra integral equation · Living sector · Green function method

References

[1] Qu Z., Jiang P. and Zhang W., Development and application of infrared thermography non-destructive testing techniques, Sensors, 20: 38-51, 2020.

[2] El Badia A. and Ha-Duong T., On an inverse source problem for the heat equation. Application to a pollution detection problem II, Journal of Inverse and Ill-Posed Problems, 10: 589-599, 2002.

[3] Orlande H., Inverse problems in heat transfer: new trends on solution methodologies and applications, Journal of Heat Transfer, 2010.

[4] Summa S., Tarabelli L., Di Perna C. and Stazi F., Data-driven automation in HVAC systems: an experimental study in a university study room, Journal of Building Engineering, 95: 110-166, 2024.

[5] Boria S., Egidi N., Fatone L., Giacomini J., Maponi P. and Piombin R., Advanced computational techniques for heat source detection, in preparation.

[6] Vladimirov V.S., Equations of mathematical physics, Marcel Dekker Inc., 1971

[7] Atkinson K.E., An introduction to numerical analysis, John Wiley and Sons, 1989.

[8] Tikhonov A. and Arsenin V., Solutions of ill-posed problems, Winston, 1977.