*VI International Conference on Mathematics and its Applications in Science and Engineering*
*(ICMASE 2025)*

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# SHARP LORDEN-TYPE BOUNDS FOR THE OVERSHOOT OF MULTIDIMENSIONAL MARTINGALES WITH APPLICATIONS TO RISK ASSESSMENT

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## ABSTRACT

**Motivation.** In risk management a loss process is monitored and a breach is declared when it first exceeds a regulatory threshold $b$. The main quantity for capital–reserve sizing is the *overshoot* $R_b$—how far the loss surpasses $b$. We give moment-based bounds on $\mathbb{E}[R_b]$ for *vector* martingales, delivering dimension-free worst-case estimates for portfolios driven by multiple risk factors.

**Model.** Let $\{M_n\}_{n\geq 0}$ be an $\mathbb{R}^d$-valued martingale with square-integrable increments $X_n = M_n - M_{n-1}$ and

$$\mu = \mathbb{E}\,\|X_1\| \;>\; 0, \qquad \sigma^2 = \operatorname{Var}\big(\|X_1\|\big).$$

For $b > 0$ define the stopping time and overshoot

$$\tau_b = \inf\{n \geq 1 : \|M_n\| > b\}, \qquad R_b = \|M_{\tau_b}\| - b.$$

**Theorem (universal overshoot bound).** If $\mathbb{E}[\tau_b] < \infty$, then for *every* $b > 0$

$$\mathbb{E}[R_b] \;\leq\; \frac{\mathbb{E}\,\|X_1\|^2}{\mu}$$

The bound depends only on the first two moments of one increment and is independent of both the dimension $d$ and the level $b$.

*Idea of proof.* Set $Y_n = \|M_n\|^2$. Martingale properties yield $\mathbb{E}Y_{\tau_b} = \mathbb{E}\tau_b\,\mathbb{E}\|X_1\|^2$. Jensen gives $\mathbb{E}\|M_{\tau_b}\| \leq \sqrt{\mathbb{E}Y_{\tau_b}}$. Wald's identity $\mathbb{E}\tau_b = (b + \mathbb{E}R_b)/\mu$ then implies the stated inequality.

**Risk-buffer corollary.** The simpler bound

$$\mathbb{E}[R_b] \;\leq\; \mu + \frac{\sigma^2}{\mu}$$

is often tighter for light-tailed increments and can be interpreted as a conservative capital add-on computable in real time without simulation.

**Implications for practice.** These moment-only formulas furnish stress-testable limits on average exceedance of solvency or liquidity thresholds by multivariate positions, avoiding dimensional blow-up.

***Keywords*** martingale overshoot · Lorden inequality · risk buffer · stopping time · dimension-free bound

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