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GALOIS BUNDLES AND AUTOMORPHISMS OF THE PRINCIPAL BUNDLE MODULI SPACE

Álvaro Antón-Sancho[1,*]

[1] *Fray Luis de León University College of Education, Catholic University of Ávila. C/Tirso de Molina, 44, 47010 Valladolid (Spain).*

ABSTRACT

Given a compact Riemann surface X and a semisimple complex Lie group G, Ramanathan [9, 10] constructed the moduli space $M(G)$ of polystable principal G-bundles over X. The study of the geometry and topology of these moduli spaces have several applications in many fields of mathematics, theoretical physics and engineering. First, the holomorphic structures admitted by a given principal G-bundle is in bijective correspondence with the Dolbeault operator space, in the spirit of the Newlander-Nirenberg theorem [11]. Indeed, the latter space is isomorphic to the space of flat connections with values in a maximal compact subgroup H of G, which links principal G-bundles to the solutions of the Yang-Mills equations of quantum field theory [3]. In addition, the moduli space $M(G)$ is isomorphic to the moduli space of representations of the fundamental group of the base curve X in H [8, 7]. Likewise, in technical, applied science and engineering areas, principal bundles are used, among other situations, in contexts of chaotic prediction in engineering problems [6]. One of the most fruitful lines of research concerning the geometry of the moduli space of principal G-bundles is the study of their automorphisms and, in particular, the identification and analysis of their fixed points subvarieties [1]. Fringuelli [4] proved that there are three families of automorphisms of $M(G)$ that generate, by composition, the whole group of automorphisms: the action of an outer automorphism of G; the action, by pull-back, of an automorphism of X; and the action, by tensor product, of line bundles parametrized by $H^1(X, Z)$, where Z denotes the center of G. The literature has extensively studied, above all, automorphisms of the first type, in different contexts: outer involutions in the moduli space of vector bundles [5] or actions of outer automorphisms of G when G is a simple group [1]. In the present work, automorphisms defined as a composition of the action of an outer automorphism of G of order 2 with the action of an involutive automorphism of the base curve X are considered. Thus, it will be assumed that G admits only one nontrivial outer automorphism σ, which has order 2 (this is the case, for example, of the special linear group) and that X is a hyperelliptic curve, so that σ acts as the hyperelliptic involution of X. Specifically, this work provides a description of the fixed points of this type of automorphisms. To do this, the notion of Galois G-bundle, introduced in [2] for the specific case of principal bundles with structure group E_6, is used. The principal G-bundles E that admit a Galois structure are exactly the fixed points of the mentioned automorphism such that the iteration of the isomorphism $E \to \sigma^*(\sigma(E))$ is the identity $E \to E$. Therefore, a generalization to semisimple Lie groups of the results proved in [2] is provided here. Specifically, for a Galois G-bundle that admits a nontrivial automorphism commuting with the Galois structure, it is provided a reduction of its structure group to the centralizer of a non-central semisimple element of G. From this, a vector form of the Galois G-bundles above is induced, given a faithful representation of G. Furthermore, the relationship that exists between the Galois G-bundles and the Galois G/Z-bundles is discussed, Z being the center of G. Specifically, sufficient conditions are provided for a G/Z-bundle Galois structure to lift a G-bundle Galois structure.

Keywords Galois bundle · Principal bundle · Automorphism · Fixed point · Lie group · Semisimple

[*] *Corresponding Author's E-mail: alvaro.anton@frayluis.com*

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