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ON FIBONACCI POLYNOMIALS AND DETERMINANTS

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ABSTRACT

The classical sequence of Fibonacci polynomials is a special case of Chebyshev polynomials and has been studied in various fields. This class of polynomials, denoted by $\{F_n(x)\}_{n\geq 0}$, is defined in [2] by the initial conditions $F_0(x) = 0$, $F_1(x) = 1$, and the recurrence relation: $F_{n+1}(x) = xF_n(x) + F_{n-1}(x),$ for $n \geq 1$, as well as by the following identity:

$$F_{n+1}(x) = E_n(x,-1) = \sum_{m=1}^{\infty} \binom{n-m}{m} x^{n-2m} = \det \begin{pmatrix} x & -1 & 0 & 0 & \cdots \\ 1 & x & -1 & 0 & \cdots \\ 0 & 1 & x & -1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & 1 & x \end{pmatrix}_{n\times n} . \quad (1)$$

The Identity (1) was derived from the second-type Dickson polynomials presented in [3]. In [2], several identities for Fibonacci polynomials were obtained using matrix theory. Moreover, various properties and generalizations of this well-known sequence can be found in the literature (see, for example, [1], [5], [4], and [6]), along with numerous applications in algebra, analysis, combinatorics, and matrix theory. For instance, in [5], a new procedure for the numerical solution of boundary value problems involving expansions in Fibonacci polynomials was introduced. The fundamental Fibonacci system, as introduced in [6], is defined by the recurrence relation

$$F_n^{(s)} = F_{n-1}^{(s)} + F_{n-2}^{(s)} + \cdots + F_{n-r}^{(s)}, \quad n \geq r, \quad (2)$$

where the initial conditions are given by $F_n^{(s)} = \delta_{s-1,n},$ for $0 \leq n \leq r-1$. The properties of the fundamental Fibonacci system and the fundamental solution were established, and new identities for generalized Fibonacci numbers were derived. Considering these previous results, this work focuses on a generalization given by the following linear difference equation of order $r \geq 2$:

$$F_n^{(s)}(x) = xF_{n-1}^{(s)}(x) + \sum_{i=1}^{r-1} F_{n-i-1}^{(s)}(x), \quad \forall n \geq r, \quad (3)$$

with initial conditions given by $F_{s-1}^{(s)}(x) = 1$, $F_n^{(s)}(x) = 0,$ for all $x \in \mathbb{R},\ 0 \leq n \neq s-1 \leq r-1.$ We refer to the class of polynomials defined by Equation (3) as the Fibonacci polynomials.

Our goal is to study Equation (3) through the fundamental Fibonacci system and Chebyshev polynomials, that is, a determinantal approach to the terms of this polynomial sequence. This method allows us to establish new properties and identities for this newly generalized class of polynomials.

Fibonacci polynomials, a special case of Chebyshev polynomials, can be expressed as determinants of structured matrices, such as Jacobi or tridiagonal matrices. These representations make explicit

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the recurrence relations between the polynomials and facilitate the analysis of their properties, as we will discuss next. Given the following result, several new identities can be derived.

Theorem 1 *Let F_r be the fundamental system of Fibonacci polynomials. Then, for $r \geq 2$, $1 \leq s \leq r-1$, and $n \geq 1$, we have:*

$$F^{(s)}_{n+r-1}(x) = det(R_{n,s}) = det \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & x & -1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & x & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & 0 \\ 1 & 1 & \cdots & 1 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \cdots & \ddots & x & -1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & x \end{pmatrix}_{n \times n}.$$

$$R_{n,s_{i1}} = \begin{cases} 1, & \textit{if } 1 \leq i \leq s, \\ 0, & \textit{otherwise.} \end{cases}$$

Keywords Chebyshev Polynomials · Generalized Fibonacci polynomials · Fundamental system

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