Computable Exponential-Type Bounds for the Expected Residual Waiting Time under Heavy-Tailed Distributions

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Abstract

The residual waiting time process $W(t)$ arises in many application areas, including queueing systems, reliability analysis, stock-control and inventory systems. However, except for certain special distributions such as the Erlang distribution, obtaining closed-form expressions for the expected residual waiting time $\mathbb{E}[W(t)]$ is generally analytically difficult. The aim of this study is to obtain valid exponential-type bounds for the expected residual waiting time $\mathbb{E}[W(t)]$. Since $\mathbb{E}[W(t)]$ and the renewal function $U(t)$ are related by $\mathbb{E}[W(t)] = \mu U(t) - t,\ \mu = \mathbb{E}[X] < \infty$, the problem can be reduced to deriving lower and upper bounds for $U(t)$. The resulting bounds obtained in this study remain valid, including heavy-tailed cases for which $\mathrm{Var}(X) = \infty$.

The $\kappa(t)$-based exponential-type bounds proposed by Chadjiconstantinidis (2024) provide sharp results for $U(t)$. However, since $\kappa(t)$ is defined by the integral root equation

$$\int_0^t e^{\kappa(t)y}\, dF(y) = 1,$$

its evaluation generally requires numerical computation in practice.

Building upon the approach of Chadjiconstantinidis (2024), this study first derives a lower bound $\kappa_{\mathrm{low}}(t)$ by using the convexity of the exponential function. An upper bound $\kappa_{\mathrm{up}}(t)$ is then obtained with the aid of the truncated second moment $m_2(t) = \mathbb{E}\left[X^2 \mathbf{1}_{\{X \le t\}}\right]$. Thus, the analytic bracketing $\kappa_{\mathrm{low}}(t) \le \kappa(t) \le \kappa_{\mathrm{up}}(t)$ is established. This interval is subsequently narrowed by the bisection method, yielding bounds for $U(t)$ whose validity is preserved, and consequently lower and upper bounds for $\mathbb{E}[W(t)]$.

The numerical results demonstrate that the $\kappa(t)$-based values reported in the literature can be approached with only a small number of iterations, for the heavy-tailed numerical examples considered in this study, while the bounding property is preserved at every step.

Keywords Residual waiting time · Renewal function · Heavy-tailed distributions · Exponential-type bounds · Bisection method

References

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