DETERMINANTAL APPROACH APPLIED TO THE NON-HOMOGENEOUS $(q, k)-$FIBONACCI-STIRLING SEQUENCE

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ABSTRACT

The sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers is defined recursively by the relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = F_1 = 1$. Among its interpretations, we can highlight that these numbers represent the count of compositions of the integer $n + 1$ with no part equal to 1 (see [2], Sequence A000045). On the other hand, the Stirling numbers of the first kind $S(n, k)$, $0 \leq k \leq n$, determine the number of permutations in the symmetric group S_n that decompose into exactly k cycles. Formulas for $S(n, k)$ in terms of partitions of a positive integer, cyclic types of a permutation, and Vieta's formula can be found, respectively, in the works of [3, 4] and [5].

In this work, we are interested in the non-homogeneous recurrence relation with variable coefficients given by:

$$P^{(k)}_{n+2}(q) = p^{(k)}_{1,n}(q)P^{(k)}_{n+1}(q) + p^{(k)}_{2,n}(q)P^{(k)}_{n}(q) + S(n + 2, k),\ 0 \leq k \leq n, \tag{1}$$

where $p^{(k)}_{1,n}(q) = 1 - q^2 + q^{2n+2k+1}$, $p^{(k)}_{2,n}(q) = q^2$ and with initial conditions $P^{(k)}_0(q) = 1$ and $P^{(k)}_1(q) = 1 + q^{2k+1}$. Note that if we consider only the homogeneous part of (1) with the initial conditions, when $q \to 1$, we obtain the sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers. Therefore, the homogeneous part of (1) is a q-analog of the sequence F_n, as corroborated in [1].

Now, if we take the vector $Y^{(k)}_n = (P^{(k)}_n(q), P^{(k)}_{n+1}(q))^T$ and $X^{(k)}_n = (0, S(n + 1, k))^T$, from the recursive process we find that Equation (1) can be represented in matrix form as

$$Y^{(k)}_{n+1} = T^{(k)}_q(n + 1)Y^{(k)}_0 + \sum_{r=1}^{n}\prod_{i=r}^{n} L^{(k)}_q(i)X^{(k)}_r + X^{(k)}_{n+1}, \tag{2}$$

where $T^{(k)}_q(n) := \prod_{h=0}^{*,n-1} L^{(k)}_q(h)$ is the second order transition matrix, $L^{(k)}_q(n)$ is the companion matrix associated to Equation (1) and $Y^{(k)}_0$ is the vector of initial conditions. Thus, in order to find explicit formulas for the entries of $T^{(k)}_q(n)$, we will consider the canonical solutions $\psi^{(0)}_n(q, k)$ and $\psi^{(1)}_n(q, k)$ of Equation (1), which are defined, for $s \in \{0, 1\}$, as $\psi^{(s)}_n(q, k) := \det \Phi^{(s)}_{n-1}$, for $n \geq 2$ and, when $n = 0, 1$, we define $\psi^{(s)}_n = 1$ for $n = 1 - s$ and 0 otherwise. The matrix $\Phi^{(s)}_{n-1}$ is the tridiagonal matrix generated by the first $n - 1$ rows and columns of the infinite matrix $\Phi^{(s)}$, given by:

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$$\Phi^{(s)} = \begin{pmatrix} p_{1+s,0}(q) & p_{2+s,1}(q) & & & \\ -1 & p_{1,1}(q) & p_{2,2}(q) & & \\ & -1 & p_{1,2}(q) & p_{2,3}(q) & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

and with $p_{m,n}(q) = 0$ whenever $m > 2$. Then, from the previous data, we are able to show that the explicit determinantal expression for the sequence $\{P_n^{(k)}(q)\}_{n\geq 0}$ is given by:

$$P_{n+2}^{(k)}(q) = (1 + q^{2k+1})\psi_{n+2}^{(0)}(q,k) + \psi_{n+2}^{(1)}(q,k) + \sum_{r=1}^{n+1} \psi_{n-r+2}^{(0,r)}(q,k)S(r+1,k), \tag{3}$$

for all $0 \leq k \leq n$. Thus, from the formulas for the homogeneous solution of (1) obtained in [1] and the formulas for $S(n,k)$ found in [3]-[5], we can establish an explicit combinatorial formula for the sequence $\{P_n^{(k)}(q)\}_{n\geq 0}$.

***K*eywords** Fibonacci sequence · Stirling numbers of the first kind · Recurrence relations · Determinantal approach

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