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COATOMS AND THE BOOLEAN IMAGE IN THE $\mathcal{R}_1$-LATTICE ON T_{01} FOR 3-VALUED LOGIC

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ABSTRACT

Let $E_3 = \{0, 1, 2\}$ and let P_3 denote the clone of all operations on E_3. By Post [1] the lattice of clones on a two-element domain is countable; by Yanov and Muchnik [2] it already has the cardinality of the continuum on any finite domain of cardinality at least three, ruling out a Post-style enumeration on E_3. A standard response, the *strengthened-closure programme*, is to enrich the clone closure by an additional invariance whose lattice of closed sets is more tractable [3,5].

We single out the value $2 \in E_3$ as *distinguished* and consider the equivalence $f \equiv_{01} g$ iff $\mathrm{ar}(f) = \mathrm{ar}(g)$ and $f^{-1}(2) = g^{-1}(2)$. The operator $\mathcal{R}_1$ is the smallest closure operator on P_3 that contains the projections, is closed under superposition and the standard arity equivalence, and additionally identifies $\equiv_{01}$-equivalent functions. In the language of matrix logics in the sense of Wójcicki [4], $\equiv_{01}$ is precisely the relation of $\mathfrak{M}$-indistinguishability for the three-valued matrix $\mathfrak{M} = (E_3, P_3, \{0, 1\})$: $\mathcal{R}_1$ records the input set producing the non-designated value while collapsing all internal differences within the designated set.

For the natural ambient class $T_{01} := \{f \in P_3 : f(\{0, 1\}^n) \subseteq \{0, 1\}\}$ — operations whose restriction to the Boolean cube is itself Boolean, see [5] — we obtain a complete classification of maximal proper $\mathcal{R}_1$-closed subclasses.

Theorem 1. *The class T_{01} has exactly three $\mathcal{R}_1$-precomplete subclasses, namely*

$$T_{01} \cap T_2, \qquad T_{01} \cap T_\sim, \qquad T_{01} \cap I_2.$$

Moreover, their pairwise intersections, together with the triple intersection and T_{01} itself, give at least eight pairwise distinct $\mathcal{R}_1$-closed subclasses, forming a sublattice in the lattice of $\mathcal{R}_1$-closed subclasses of T_{01}.

The $\sim$-preserving class $T_\sim$ admits a sharper description. Let $\pi\colon E_3 \to E_2$ be the projection $\pi(0) = \pi(1) = 0$, $\pi(2) = 1$, extended coordinatewise. For $f \in T_\sim$, $\sim$-preservation forces a unique $\widetilde{f} \in P_2$ with $\widetilde{f} \circ \pi = \pi \circ f$, the *Boolean image* of f.

Theorem 2. *The map $[f]_{\equiv_{01}} \mapsto \widetilde{f}$ is a clone isomorphism between the quotient $(T_\sim, \mathcal{R}_1)/\equiv_{01}$ and the standard Boolean clone (P_2, C).*

Corollary 1. *The lattice of $\mathcal{R}_1$-closed subclasses of $T_\sim$ is isomorphic to Post's lattice [1], and is therefore countably infinite.*

The two precomplete subclasses $T_{01} \cap T_2$ and $T_{01} \cap I_2$ contrast sharply with $T_\sim$: their $\mathcal{R}_1$-closed subclasses encode genuinely three-valued combinatorics through the structure of $f^{-1}(2) \subseteq E_3^n$, and need not be countable. The main theorem can therefore be read as identifying $T_\sim$ as the unique "Boolean" coatom in the $\mathcal{R}_1$-lattice on T_{01}. Symmetric variants of the classification for the value-layers T_{02} and T_{12}, obtained by relabelling the distinguished value, are also recorded; building on

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the first author's earlier work [6] on lattice structure of 01-functions, this completes a triad of layer-wise classifications under $\mathcal{R}_1$.

***K*eywords** three-valued logic · matrix logics · designated values · clone theory · closure operators · precomplete classes

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