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THE FUNDAMENTAL SYSTEM OF PELL POLYNOMIALS AND THE COMPANION MATRIX

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ABSTRACT

The well-known Pell polynomial sequence, denoted by $\{P_n(x)\}_{n\geq 0}$ is defined by the initial conditions $P_0(x) = 0$, $P_1(x) = 1$, and the recurrence relation $P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x),$ for $n \geq 1$, as presented in the literature (see, for example, [1], [4],[5]). Generalizations of this sequence can be found in the literature (see, for instance, [3], [8], [5]), along with numerous applications in algebra, analysis, combinatorics, and matrix theory. In [8], the Pell fundamental system is defined by the recurrence relation $P_n^{(s)} = P_{n-1}^{(s)} + P_{n-2}^{(s)} + \cdots + P_{n-r}^{(s)}, \quad n \geq r$, with initial conditions given by $P_n^{(s)} = \delta_{s-1,n}, \quad 0 \leq n \leq r-1$. Thus, considering these results, in this work, we focus on a generalization given by the following linear difference equation of order $r \geq 2$:

$$P_n(x) = 2xP_{n-1}(x) + \sum_{i=1}^{r-1} P_{n-i-1}(x), \quad \forall n \geq r, \tag{1}$$

with the initial conditions $P_0(x), P_1(x), \ldots, P_{r-1}(x)$. We call the class of polynomials given by Equation (1) the Pell polynomials. Our goal is to study Equation (1) via Pell fundamental system and the companion matrix associated. Considering the properties of the fundamental system, that is, the relationship between the sequences of Pell polynomials, as well as the connection of this family of sequences with the power of the associated companion matrix, several new results are derived, that is, new results of generalizations of properties already known for Pell polynomials or even Pell numbers.

Based on the connection between Markov chains and sequences of linear recurrence relations presented in [6] and results presented in [2], several combinatorial results for generalized Pell polynomials are derived.

In [6] we have that for $n > m \geq r$, the number $\rho(n, m)$ is a probability, given by:

$$\rho(n,m) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-m} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!} a_0^{k_0} \ldots a_r^{k_{r-1}}. \tag{2}$$

In [2] we have that the entries of the power of an associated companion matrix can be written as:

$$c_{ij}^{(n)} = \sum_{k_1+2k_1+\ldots+mk_m=n-i+j} \frac{k_j+\cdots+k_m}{k_1+\cdots+k_m} \frac{(k_1+\cdots+k_m)!}{k_1!k_1!\ldots k_m!} a_1^{k_1} \ldots a_m^{k_m}, \tag{3}$$

Thus, considering these results, we can write the entries of the companion matrix associated with the Pell polynomials and obtain the following expression.

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$$P_n^{(r)}(x) = \rho(n+1, r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n+1-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!}(2x)^{k_0}, \quad n \geq r, \quad (4)$$

where $\rho(1, r) = \cdots = \rho(r-1, r) = 0$ and $\rho(r, r) = 1$.

***K*eywords** Generalized Pell polynomials · Fundamental system · combinatorial identities · Companion Matrix

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