LINEAR RECURRENCES AND DECOMPOSABLE FORMS

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ABSTRACT

Let $k \geq 2$ denote an integer. Assume that there are given k distinct, linearly independent, homogeneous linear recurrences of order k satisfying the same recurrence relation. These recurrences are related to a decomposable form of degree k, and there is a general identity with a suitable exponential expression depending on the recurrences. This identity is a common and very broad generalization of several known identities. If the recurrences are integer sequences, then the diophantine equation associated to the decomposable form and the exponential term has infinitely many integer solutions generated by the terms of the recurrences. A method for the complete factorization of the decomposable form can be given. Both the form and its decomposition are found if $k = 2$, and a few examples for $k = 3$ are shown.

An example is provided here with two binary recurrences. Let $G_0 = 1$, $G_1 = 2$, $H_0 = 3$, $H_1 = 4$, and both sequences (G_n) and (H_n) satisfy the recurrence rule $x_n = 4x_{n-1} - x_{n-2}$. Then the identity

$$23G_n^2 - 18G_nH_n + 3H_n^2 = -4 \tag{1}$$

holds, where the left-hand side is the corresponding (quadratic) form depends on the initial values and the coefficients of the recurrences. Equality (1) resembles us the well-known identity

$$L_n^2 - 5F_n^2 = \pm 4,$$

where (F_n) denotes the Fibonacci sequence and (L_n) is its associate or companion sequence (i.e. sequence of Lucas numbers). But in case of (1) we had no prescribed connection (like associate sequence) between (G_n) and (H_n), only the common recurrence rule was fixed. It turned out that there exists a similar identity for any two recurrences with property $x_n = 4x_{n-1} - x_{n-2}$, and this identity can be given explicitly.

***K*eywords** linear recurrence · decomposable form · general identity · diophantine equation.

References

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