A BI-PERIODIC q-ANALOGUE FOR THE POWERS-OF-TWO SEQUENCE: COMBINATORIAL AND DETERMINANTAL APPROACHES

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ABSTRACT

A very important sequence in the area of discrete mathematics as well as in Number Theory is given by the recurrence relation $P_n = 2P_{n-1}$, with $P_0 = 1$. It refers to the powers of the integer 2, since $P_n = 2^n$ for all $n \geq 0$. This sequence is closely linked to Set Theory, as it represents the number of subsets of a set with n elements ([3], p. 8). By the other hand, this sequence represents the number of compositions of the integer n (see [4], Eq. (39)). Furthermore, several other interpretations are associated with the sequence $\{P_n\}_{n\geq 0}$, as can be seen in [7].

However, in this work, we focus on a q-analogue of the sequence $\{P_n\}_{n\geq 0}$, which is a polynomial in the indeterminate q that generalizes the sequence under consideration. That way, consider the following bi-periodic sequence with variable coefficients,

$$P_n(q) = \begin{cases} (1+q)P_{n-1}(q) + (q^{2n} - q)P_{n-2}(q), & \text{if } n \text{ is even} \\ (1+q)P_{n-1}(q) + (q^{2n-1} - q)P_{n-2}(q), & \text{if } n \text{ is odd} \end{cases}, \tag{1}$$

and with initial conditions $P_0(q) = 1$ and $P_1(q) = 1 + q$. Clearly, when $q = 1$, we have $P_n(1) = P_n = 2^n$, regardless of the parity of the sequence index. Although Equation (1) yields a bi-periodic polynomial sequence, we can represent $\{P_n(q)\}_{n\geq 0}$ as

$$P_n(q) = (1+q)P_{n-1}(q) + (q^{2n-\xi_n} - q)P_{n-2}(q),$$

where $\xi_n = \dfrac{1-(-1)^n}{2} = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$ is the parity function defined by [2]. Now, observe that, if we substract $P_{n+1}(q) - P_n(q)$ when n is even and odd, respectively, we obtain

$$\begin{aligned} P_{n+1}(q) &= (2+q)P_n(q) - (1 + 2q - q^{2n+1})P_{n-1}(q) + (q - q^{2n})P_{n-2}(q), & (2) \\ P_{n+1}(q) &= (2+q)P_n(q) - (1 + 2q - q^{2n+2})P_{n-1}(q) + (q - q^{2n-1})P_{n-2}(q). & (3) \end{aligned}$$

Thus, Equations (2) and (3) allows us to establish that the sequence $\{P_n(q)\}_{n\geq 0}$ satisfy the following property

$$P_{n+1}(q) = (2+q)P_n(q) - (1 + 2q - q^{2n-\xi_{n+1}+2})P_{n-1}(q) + (q - q^{2n-\xi_n})P_{n-2}(q), \; n \geq 2. \tag{4}$$

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Consider the vector $Y_n = (P_n(q), P_{n+1}(q))^T$. Then, from the recursive process we find that Equation (1) can be represented in matrix form as $Y_{n+1} = T_q(n+1)Y_0$, where $T_q(n) := \prod_{h=0}^{*,n-1} L_q(h)$ is the second order transition matrix, $L_q(n)$ is the companion matrix associated to Equation (1) and Y_0 is the vector of initial conditions. Thus, in order to find explicit formulas for the entries of $T_q(n)$, we will consider the canonical solutions $\psi_n^{(0)}(q)$ and $\psi_n^{(1)}(q)$ of Equation (1), which are defined, for $s \in \{0,1\}$, as $\psi_n^{(s)}(q) := \det \Phi_{n-1}^{(s)}$, for $n \geq 2$ and, when $n = 0, 1$, we define $\psi_n^{(s)}(q) = 1$ for $n = 1 - s$ and 0 otherwise.

The matrix $\Phi_{n-1}^{(s)}$ is the tridiagonal matrix generated by the first $n-1$ rows and columns of the infinite matrix $\Phi^{(s)}$, namely

$$\Phi^{(s)} = \begin{pmatrix} p_{1+s,0}(q) & p_{2+s,1}(q) & & & \\ -1 & p_{1,1}(q) & p_{2,2}(q) & & \\ & -1 & p_{1,2}(q) & p_{2,3}(q) & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $p_{1,n}(q) = 1+q$, $p_{2,n}(q) = q^{2n-\xi_n+4} - q$ and with $p_{m,n}(q) = 0$ whenever $m > 2$. Then, from the previous data, we are able to show that the explicit determinantal expression for the sequence $\{P_n(q)\}_{n\geq 0}$ is given by

$$P_n(q) = (1+q)\psi_n^{(0)}(q) + \psi_n^{(1)}(q), \tag{5}$$

for all $n \geq 0$. Thus, from the formulas for the determinants $\{\psi_n^{(s)}(q)\}_{n\geq 0}$ obtained in [1], we can establish an explicit formula for Equation (5) in terms of nested sums. By the other hand, the sequence $\{P_n(q)\}_{n\geq 0}$ offer interesting combinatorial interpretations for some series-product identities, listed by Slater in [6] and also by Santos in [5].

Keywords Powers-of-two sequence · Recurrence relations · Variable coefficients · Bi-periodic sequences · Determinantal approach

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