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PRINCIPAL BUNDLES WITH PRESCRIBED MONODROMY AND APPLICATIONS TO TOPOLOGICAL CONTROL SYSTEMS

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ABSTRACT

Principal bundles with a complex reductive structure group are geometric objects of great relevance in mathematical research and provide numerous applications [1]. The theory of principal bundles with prescribed monodromy, based on the theory of principal bundles to which an additional structure is included, represents a significant area of research at the intersection of differential geometry and algebraic topology, with applications in robotic navigation and control systems. The present work makes original contributions to this field, building upon established foundations in the geometric theory of bundles [2] and representation spaces [7]. Operating within the context of compact Riemann surfaces with punctures [11], this research develops a novel characterization of principal G-bundles for complex reductive Lie groups equipped with connections having prescribed monodromy around specified points. This approach extends classical results from Simpson [14] on moduli spaces of representations to new contexts with direct applications to engineering problems.

The core innovation of the present research lies in establishing a precise bijective correspondence between character varieties and moduli spaces of principal bundles with prescribed monodromy. Explicit dimensional formulas are provided that quantify how the topological complexity of the underlying surface, the structure of the Lie group, and the nature of the prescribed monodromy classes interact to determine the global geometry of the moduli space. These formulas reveal that, for a semisimple Lie group G, the dimension depends on both the genus of the surface and the dimensions of the conjugacy classes involved. This provides new insights into the geometric structure of these spaces beyond what was previously understood through the work of Hitchin [8] and Boalch [4], and further developed in recent research [5]. This study also offers a detailed analysis of the singularity structure of these moduli spaces, connecting them to bundles with non-trivial automorphism groups in a way that extends traditional approaches from geometric invariant theory [12].

The most significant contribution of this work is the development of a geometric framework for applying the theory of principal bundles with prescribed monodromy to control systems operating in topologically complex environments. Thus, by reformulating the navigation problem for robotic systems [13] as a question about connections on principal bundles over punctured surfaces, the research connects geometric structures on principal bundles and engineering challenges. This approach builds upon but substantially extends the geometric formulation of robot navigation introduced by Koditschek and Rimon [9], providing a more sophisticated treatment of the topological constraints that arise in obstacle avoidance problems. We demonstrate how the local geometry of the moduli space can be interpreted as a measure of the sensitivity of control strategies to perturbations, offering a precise way to quantify robustness that goes beyond the algorithmic approaches typically found in works like that of LaValle [10].

Another original contribution is the characterization of when two control strategies can be continuously deformed into one another while maintaining obstacle avoidance constraints. By identifying

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the obstruction to such deformations as an element of a specific cohomology group, the work establishes a topological classification of control strategies that provides new insights into the fundamental limitations of control system design. This approach connects to recent developments in applied algebraic topology for dynamical systems [3].
The methodology used in this research builds bridges between the algebraic geometric theory of principal bundles and character varieties and engineering concerns in control theory. This is in line with certain emerging approaches in computational topology for data analysis [6] but with a focus on continuous dynamical systems rather than discrete data structures.

***K*eywords** Principal G-bundles · Prescribed monodromy · Character varieties · Moduli spaces · Topological control theory

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