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THE ISOPERIMETRIC PROBLEM FOR HYPERIDEAL POLYHEDRA

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ABSTRACT

The isoperimetric problem for polyhedra originates from a question posed by Jakob Steiner in 1842: within its combinatorial class and for a fixed surface area, does a regular Euclidean polyhedron maximize volume? This question has been answered affirmatively for the regular tetrahedron, the regular octahedron, the cube, and the regular dodecahedron.

A point in the hyperbolic space $\mathbb{H}^3$ is classified as type 1, 0, or -1 according to whether it lies inside $\mathbb{H}^3$, on the sphere at infinity, or beyond the sphere at infinity, respectively. A *hyperideal polyhedron* is a polyhedron whose vertices are all of type 0 or -1. The isoperimetric problem can likewise be formulated for hyperideal polyhedra. Frigerio and Moraschini [2] solved the isoperimetric problem for truncated hyperideal tetrahedra.

In this paper, we study hyperideal polyhedra whose vertices are all of type -1, with every vertex truncated. We establish the following result:

A convex hyperideal polyhedron P_0 with all vertices of type -1 and equal edge lengths uniquely maximizes the volume among all convex hyperideal polyhedra with vertices of type -1 sharing the same combinatorial type and surface area as P_0.

Its proof relies on two fundamental properties of hyperideal polyhedra: the convexity of the space of dihedral angles, established by Bao and Bonahon [1], and the strict concavity of the volume function, established by Schlenker [3, 4].

As a consequence of this theorem, we obtain infinitely many hyperideal polyhedra that solve the isoperimetric problem within their combinatorial classes. The first family of such examples consists of hyperideal polyhedra with equal edge lengths constructed from Euclidean convex polyhedra whose faces are regular polygons.

The second family consists of hyperideal polyhedra with equal edge lengths that have no Euclidean counterparts.

Keywords isoperimetric problem · hyperideal polyhedra · hyperbolic volume · convex uniform polyhedra · Johnson solids

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