

ON DUCCI SEQUENCES WITH LEONARDO NUMBERS AND THEIR MATRIX REPRESENTATIONS: APPLICATIONS

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ABSTRACT

Leonardo numbers form a Fibonacci-related integer sequence that has recently attracted considerable attention because of its rich algebraic structure and its links to combinatorics, coding theory, and matrix analysis. In this study, we examine the n -tuples $D_{Le} = (Le_1, Le_2, \dots, Le_n)$ and $D_{GLE} = (GLE_1, GLE_2, \dots, GLE_n)$, formed respectively from the Leonardo numbers Le_n and the Gaussian Leonardo numbers GLE_n , and study their behaviour under the Ducci map $D : \mathbb{C}^n \rightarrow \mathbb{C}^n$, defined by taking absolute differences of cyclically adjacent entries. We derive the first and second Ducci iterates D_{Le} , D_{Le}^2 and D_{GLE} , D_{GLE}^2 in closed form, and use these iterates to construct circulant matrices whose entries are Leonardo and Gaussian Leonardo numbers. Building on this construction, we design a dual-layer coding and decoding algorithm: a plaintext message matrix is first encoded by left multiplication with a Leonardo (or Gaussian Leonardo) circulant matrix generated under the Ducci map, and the resulting matrix is then perturbed by the incidence matrix of an auxiliary labelled graph, so that the overall security depends jointly on the algebraic structure of the circulant matrix and the combinatorial choice of the graph. Explicit numerical examples, including full encoding and decoding of sample text messages, illustrate the practicality of the scheme. In the second part of the study, we investigate determinantal and permanental representations associated with Leonardo numbers under the Ducci map. Using the characteristic polynomial of an associated tridiagonal matrix and its connection to Chebyshev polynomials of the second kind, we obtain a closed-form expression for the relevant determinant in terms of Fibonacci numbers. We further apply the contraction method for permanents to a Hessenberg-type matrix and establish a permanental identity involving Leonardo and Fibonacci numbers. A Maple 13 source code implementing the contraction algorithm is also provided so that the theoretical results can be verified computationally. Overall, the paper contributes new links between Ducci sequences, Leonardo-type number families, circulant and Hessenberg matrices, and offers a graph-based cryptographic application of independent interest.

Keywords Leonardo numbers · Gaussian Leonardo numbers · Ducci map · circulant matrices · coding theory

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