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# On solutions of a third order linear difference equation with variable coefficients. Applications.

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## Abstract

In this article, we explore the determinantal and combinatorial approaches to third order linear difference equations with variable coefficients. We present and demonstrate some of its properties. Furthermore, as an application, we examine the generalized Jacobsthal sequence and the sequence of powers of two related to the Rogers-Ramanujam type identities establishing new results for these polynomial sequences.

The sequence of Jacobsthal numbers is usually represented by the recurrence relation given by $J_n = J_{n-1} + 2J_{n-2}$, with initial conditions $J_0 = 0$ and $J_1 = 1$. Applications of this recurrence can be seen in several works, such as the data in [1, 8, 5] where combinatorial and analytical interpretations are offered for this sequence of integers. Another much studied sequence is given by the recurrence relation $P_n = 2P_{n-1}$, with $P_0 = 1$. It refers to the powers of the integer 2, since $P_n = 2^n$ for all $n \geq 0$. This sequence is closely linked to Set Theory, as it represents the number of subsets of a set with $n$ elements. Furthermore, several other interpretations are associated with this sequence, as can be seen in [7]. More specifically, as applications, we explore the $q$-analogues of the sequences $J_n$ and $P_n$, denoted here by $\{J_n(q)\}_{n\geq 0}$ and $\{P_n(q)\}_{n\geq 0}$.

A $q$-analogue is a polynomial in the indeterminate $q$ that generalizes certain sequences of integers. The $q$-Jacobsthal $J_n(q)$ ($q-$ analogue for $J_n$) satisfies the following recurrence relation of order 3 with variable coefficients

$$J_{n+2}(q) = J_{n+1}(q) + (q + q^n)J_n(q) + (q^n - q)J_{n-1}(q) \tag{1}$$

with initial conditions $J_0(q) = 1 = J_1(q)$ and $J_2(q) = 1 + q + q^2$. This generalization was introduced by Santos in [3] and worked on by Craveiro in [2]. Note that, when $q = 1$, then $J_{n+2}(1)$ is nothing more than the sequence of Jacobsthal numbers.

The sequence $q-$powers of 2 was introduced by Sills in [4] and satisfies the following difference equation

$$P_{n+2}(q) = (1 + q)P_{n+1}(q) + (q^{n+3} - q)P_n(q) + (q^{2n+4} - q^{n+3})P_{n-1}(q), \tag{2}$$

with $P_0(q) = 1, P_1(q) = 1 + q$ and $P_2(q) = 1 + q + q^2 + q^3$. For $q = 1$ we have $P_{n+2}(1) = 2^{n+2}$. The sequences given by Equations (1) and (2) offer interesting combinatorial interpretations for some series-product identities, listed by Slater in [6] and also by Sills in [4]. By satisfying difference equations, we can use various techniques and methods to find new expressions for the $q-$analogues.

***K*eywords** Third order linear difference equations · Determinantal approach · Combinatorial approach · Jacobsthal sequence · Powers of two sequence · Rogers-Ramanujan Identities

## References

[1] Aydin F. T., On generalizations of the Jacobsthal sequence, Notes on Number Theory and Discrete Mathematics, 2018.

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[2] Craveiro I. M., Extensões e Interpretações Combinatórias para os Números de Fibonacci, Pell e Jacobsthal, Ph.D. Thesis, UNICAMP, Campinas, São Paulo, Brazil, 2004.

[3] Santos J. P. O., Computer Algebra and Identities of the Rogers–Ramanujan Type, Ph.D. Thesis, Pennsylvania State University, United States, 1991.

[4] Sills A. V, Computer Assisted Explorations of Rogers-Ramanujam Type Identities, Ph.D. Thesis, University Park, Pennsylvania, United States, 2002.

[5] Silva E. V. P. and Andrade C. P. and Silva K. P. C, Interpretações Combinatórias para os Números de Jacobsthal e Jacobsthal-Lucas Generalizados via Ladrilhamentos, Proceeding Series of the Brazilian Society of Applied and Computational Mathematics, 2015.

[6] Slater L. J., Further identities of the Rogers–Ramanujan Type, Proceedings of the London Mathematical Society, 1952.

[7] Sloane N. J. A., Sequence A000079 - Powers of 2, The On-line Encyclopedia of Integer Sequences, 2024

[8] Yilmaz F. and Bozkurt D., The Generalized Order-$k$ Jacobsthal Numbers, Int. J. Contemp. Math. Sciences, 2009.